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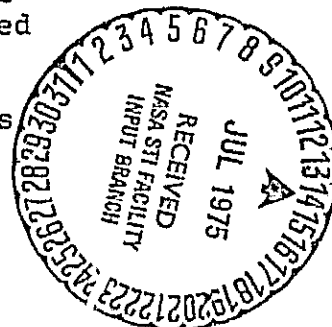
THE THEORY OF ENCELADUS AND DIONE --

AN APPLICATION OF COMPUTERIZED ALGEBRA IN DYNAMICAL ASTRONOMY

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The orbits of the satellites of the outer planets are poorly known, due to lack of attention over the past half century. We have been developing a new theory of Saturn's satellites Enceladus and Dione which is literal (all constants of integration appear explicitly), canonically invariant (the Hori-Lie method is used), and which correctly handles the eccentricity-type resonance between the two satellites. The algebraic manipulations are being performed using the TRIGMAN formula manipulation language, and the programs have been developed so that with minor modifications they can be used on the Mimas-Tethys and Titan-Hyperion systems.



INTRODUCTION

Our current knowledge about the orbits of the outer planets' satellites is quite poor. The theories now in use date back to the early part of this century, and no longer provide accurate ephemerides. At the same time, these satellites have been virtually unobserved for the better part of this century. As a result, it has been estimated by Mulholland¹ that the errors in the predicted coordinates may range into the tens of thousands of kilometers.

A primary purpose of the upcoming missions to the outer planets will be to determine improved positions for these satellites, in addition to making physical observations. It is therefore very important that their orbits be

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rediscussed in the light of modern as well as older observations. The re-observation of the satellites is already well under way at McDonald Observatory², and to go along with it, a new theoretical discussion of several of the satellites of Saturn has been undertaken. The system Enceladus-Dione is the first for which results have been obtained, and it is on this system that we report in this paper.

REMARKS ON PERTURBATION THEORY

Traditionally, there have been two principal approaches to solving perturbed motion in Dynamical Astronomy. One, numerical integration of the equations of motion, has much to commend it, and in fact, is the usual approach these days. It is straightforward and easy to implement, and can model a wide variety of situations accurately. The other approach, the so-called "literal" method, attempts to produce explicit formulae for the determination of a body's position at any arbitrary time. It is difficult to implement, as it requires special series manipulation programs -- programs which are very complicated to write and which require unfamiliar techniques in their construction. In addition, numerous technical problems arise in practice, the most common being the ease with which it is possible to build up very long expressions which threaten to overrun the entire core memory available.

Despite these disadvantages of the literal method, there are several important advantages. One is that the formulae may be good for very long periods of time, and it takes no more effort to obtain a prediction for a time far removed from the epoch of the theory than it takes to determine a prediction for a time near the epoch. With the numerical approach this is not so. A second advantage is that the constants of integration, and other parameters of the theory (e.g., the masses and oblateness parameters) can be made to appear explicitly in the formulae. This means that the theory can be used even if better values of the parameters are determined. Yet a third advantage is that each particular term in a formula arises in a predictable way and has a predictable effect on the motion of the bodies. This gives one deeper insight into the problem, particularly when resonances are of importance (as they are in this case). Thus, there are good reasons

for the interest which has been shown by celestial mechanicians in the recent progress which has been made in the area of computerized algebra for dynamical astronomy.

SYSTEMS FOR COMPUTERIZED ALGEBRA

In order to see what is presently possible, let us look a little more deeply into the kinds of processors which are currently available. We will concentrate on the Poisson series processor, which is the most commonly available. Such programs have been written by Barton³, Broucke⁴, Brumberg⁵, Chapront et al.⁶, Jefferys⁷ and Rom⁸. With several variations on a theme, they are capable of adding, subtracting, multiplying, differentiating, integrating, and performing certain substitution operations on series of the general form of Eq. (1):

$$S = \sum_k \{ C_k(x) \cos(k \cdot y) + S_k(x) \sin(k \cdot y) \}, \quad (1)$$

where $k = (k_1, \dots, k_n)$ is a vector of integers (positive, negative and zero), $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ are vectors of polynomial variables and trigonometric variables, respectively, $C_k(x)$ and $S_k(x)$ are polynomials in the x variables having rational or real coefficients (depending on the system), and $k \cdot y = k_1 y_1 + \dots + k_n y_n$ is the dot product. Thus a Poisson series is a multiple Fourier series in a linear combination (with integer multipliers) of trigonometric variables, with its coefficients drawn from a ring of polynomials in several independent variables.

Series of the form of Eq. (1) are of particular importance in dynamical astronomy because the motion of bodies in the solar system is in general quasiperiodic, and can be expressed in the form of Eq. (1) provided that the angles are linear functions of the time (with, in general, frequencies that are not rational ratios of each other). More generally, one can write the equations of motion in the form

$$\begin{aligned} \dot{x} &= \epsilon f(x, y) \\ \dot{y} &= v + \epsilon g(x, y) \end{aligned} \quad (2)$$

where the components of the vector functions f and g are Poisson series, v is the vector of the mean motions of the angles, and ϵ is a small

quantity. To first order we can write down the formal solution of these equations as

$$\begin{aligned}x &= x_0 + \epsilon \int f(x_0, y_0 + vt) dt, \\y &= y_0 + vt + \epsilon \int g(x_0, y_0 + vt) dt,\end{aligned}\tag{3}$$

calculations which can easily be performed with any of the Poisson series processors currently available.

RESONANCES

A difficulty arises - one which is of great importance in the case of Enceladus and Dione - whenever there exists a nearly exact relationship between the components of the vector v such that, for some vector of integers j , one has $v \cdot j \approx 0$. Since, in general, all possible vectors of integers j are ~~in~~ in fact represented in the series f and g , and since any real number can be approximated arbitrarily closely by a ratio of integers, this situation (as has been known for a long time) seriously threatens the convergence of Eqs. (3) and their higher order approximations. What happens is that in taking the integrals in Eqs. (3), divisors of the form $v \cdot j$ appear. Since these divisors are dense on the real line, and in particular possess 0 as a limit point, the convergence of the series is in question.

It is usual to ignore this problem as being of little practical interest (although of great theoretical interest), unless one of the terms having such a "small divisor" also happens to have a large coefficient. This is indeed the case for Enceladus and Dione, and is due to the fact that the mean motion of Enceladus is almost exactly twice that of Dione. In fact, there are two terms which are greatly amplified by the presence of a small divisor: one is the term in the angular argument $\chi = \psi - 2\psi' + \varpi$, where ψ is the longitude of Enceladus, ϖ is the longitude of its perisaturnium, and the primed variables represent the same quantities for Dione (Fig. 1). This argument has a mean motion of 30° per year, while most terms have mean motions which are over 1000 times as great. As a result, this term is amplified in Eqs. (3) by a factor which is over 1000 times greater than that for a typical term. In addition, the unamplified coefficient of this

term is one of the largest in the entire series.

An even more serious problem is posed by the term in the argument $\xi = \psi - 2\psi' + \omega$, for which the mean motion averages out to precisely zero, a situation which Garfinkel⁹ has called "deep resonance." The presence of an exact resonance such as this one causes the character of the solution to change completely. The angle θ , instead of either increasing or decreasing at a more or less constant, albeit slow rate, librates, or oscillates about some mean value (here, 0°). The reason for this behavior is that the phase plane turns out, in the lowest order, to be the same as that for a nonlinear pendulum (Fig. 2). Because of this, a different approach to this problem must be taken than the naive one indicated in Eqs. (3).

DESCRIPTION OF THE PRESENT THEORY

The detailed equations employed in the present theory will be published elsewhere¹⁰, and therefore will not be given here. Instead, we shall try to give an overall view of the theory and point out some of its more interesting features.

Our approach has been to make use of one of the recently developed Lie series methods for canonical perturbation theory (since the present problem is conservative and can be derived from a Hamiltonian). The Hori-Lie method¹¹ that we have used has a number of advantages from our point of view:

- (1) It is canonically invariant, so that the theory, once completed, is valid for any coordinate system. The reason for this is that the Poisson bracket, on which the method is based, is a coordinate-independent notion.
- (2) It is readily adapted for systematic calculation by machine.
- (3) The perturbations in any desired quantity, including directly observed quantities such as the position or velocity vector, are obtained by application of a simple and uniform formula which is readily programmed.
- (4) Perturbations are derived from the reduced Hamiltonian and only one other single function, the determining function.

Since the actual coordinate system chosen is of little importance in applying

the method, it can be chosen for convenience. We have chosen to work in variables closely related to the classical elements of semimajor axis, eccentricity and sine inclination, since it is relatively straightforward to write down the defining formulas for the disturbing function in terms of them. One modification we have adopted is to use as our actual variables, deviations from nominal values; the nominal values have been adopted from Struve's theory. The advantage of this approach is that only low order polynomials in the deviations need to be retained in the development of the Hamiltonian in order to retain sufficient accuracy. Thus the theory can be made more compact.

The first, and most difficult task is to develop the term in the Hamiltonian that describes the interaction between the two satellites. Because the ratio of the semimajor axes of the two satellites is so large (about 0.63), the series for this term converges very slowly. It is relatively straightforward to obtain an expression for the square of the distance between the two bodies, and a very short series results. However, the Newtonian interaction is proportional to the inverse of the distance, and is developed in powers of the ratio 0.63, which only slowly approaches zero.

Many methods have been developed for performing the calculation, most of them prior to the advent of modern Poisson series processors. Our method, which harks back to one described by Brouwer and Clemence¹³, has been simply to raise the expression for the square of the mutual distance to the inverse one-half power. We have employed for this an iterative technique advocated by Broucke¹⁴ which converges quadratically, i.e., the error at each step is of the order of the square of the error at the preceding step. This method has been very satisfactory and converges with great rapidity. The rest of the Hamiltonian is very simple to calculate, having none of the problems of convergence associated with the above term. This includes the "indirect" term in the Newtonian potential which arises from choosing the origin of the system at the center of Saturn, and also the terms which arise from Saturn's gravitational quadrupole moment.

Our next step is to perform a coordinate transformation, using the Hori-Lie method, to eliminate from the Hamiltonian all terms in angular combinations

which are not simple multiples of the resonant argument θ . This includes the terms in the shallowly resonant argument χ , which, although they are considerably amplified and contribute significantly to the solution, nevertheless do not change the actual character of the solution as does the argument θ . A determining function is obtained which allows the calculation of the effects of all the angular arguments except θ on the motion of the two satellites. We have also programmed a subroutine which allows the perturbations in any desired quantity to be determined, insofar as they are affected by these angular arguments.

The remaining system, which contains the deeply resonant argument θ , must now be solved. Since this is the only angular argument remaining, the system has effectively been reduced to one degree of freedom. Looking again at Fig. 2, it can be seen that the motion in θ and e is basically oscillatory in nature. In the actual case, the amplitude of the motion is quite small, only about $14'$ of arc in θ , so that it is not even necessary to worry about the nonlinear nature of the pendulum presented when the amplitude is large. In the latter case, elliptic integrals are required to get the solution; but in our case, it is sufficient to approximate the system with a linear harmonic oscillator. This we have done by introducing an appropriate new set of canonical elements, centered on the origin in Fig. 2, such that in these coordinates the equations of motion are reduced to an (amplitude dependent) harmonic oscillator. The solution to this being well-known, we have therefore arrived at the complete solution of the problem.

FINAL FORM OF THE THEORY

In this form, the theory is not immediately useful, particularly if one is lacking a Poisson series processor. Therefore we have employed another method in order to make the theory accessible to the "outside world." Since FORTRAN is nearly ubiquitous as a programming language, we have chosen to have our Poisson series processor punch out FORTRAN-compatible expressions for each of the formulae required to obtain perturbations in any desired quantity. Thus, given numerical values of the various parameters in the theory, these expressions can be used to compute numerical values of any

desired quantity at any time.

We are in the process of setting up a series of subroutines for the computation of all the necessary quantities. As can be imagined, many of the expressions are quite long and involved, so a fair amount of machine time is consumed in their evaluation. However, modern machines are really quite fast, and experience has shown that the time required for a single evaluation is not prohibitive (amounting to many evaluations per second). Thus, our method is quite practical. Furthermore, anyone who has access to a competent FORTRAN compiler will be able to use our theory.

At the present time, we are setting up programs for the comparison of our literal theory with an ephemeris generated by numerical integration, as a check on the accuracy of the theory. At the same time, we are programming a set of subroutines to obtain definitive values for the parameters of the theory (constants of integration, masses and oblateness parameters) by a differential correction procedure. Since the data are still being gathered for this step, we are checking this phase also against numerical integrations, but eventually we expect to use both the old and new observations to produce a definitive theory. We will be greatly helped in this step by the recent publication by Pierce¹⁵ of a bibliography of observational material on the satellites of Saturn.

Eventually it is hoped to include observations of the satellites from the space missions themselves. To this end, we are working for an ultimate predictive capability for the theory of ± 1 km.

Finally, we mention that the programs we have written are adaptable to more problems than the Enceladus-Dione case. Both the Mimas-Tethys and Titan-Hyperion systems also have strong resonances (involving different angular arguments). We have attempted to write the programs so that with minor changes they can be applied to these systems as well.

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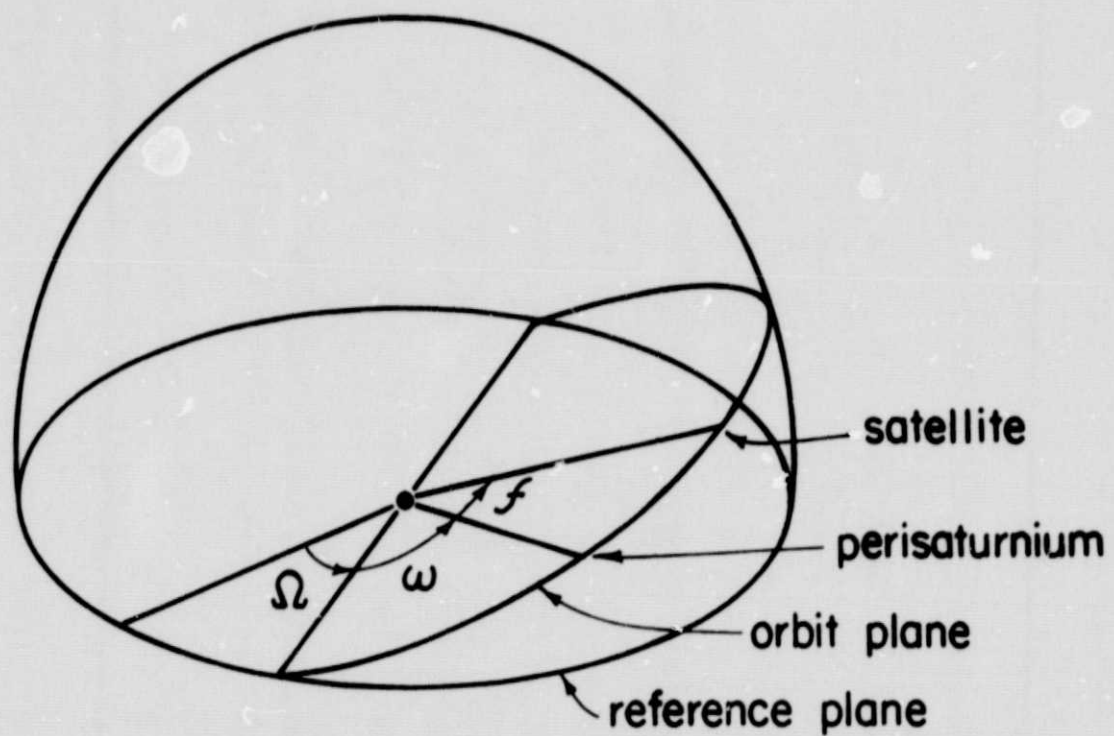
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FIGURE CAPTIONS

Figure 1: Geometry of the Orbits

Figure 2: Resonance Phase Plane



$$\varpi = \Omega + \omega$$

$$\psi = \Omega + \omega + f$$

